

FLAT CYCLES IN THE HOMOLOGY OF $\Gamma \backslash \mathrm{SL}_m \mathbb{R} / \mathrm{SO}(m)$

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ABSTRACT. In this paper we show that flat $(m-1)$ -dimensional tori give nontrivial rational homology cycles in congruence covers of the locally symmetric space $\mathrm{SL}_m \mathbb{Z} \backslash \mathrm{SL}_m \mathbb{R} / \mathrm{SO}(m)$. We also show that the dimension of the subspace of $H_{m-1}(\Gamma \backslash \mathrm{SL}_m \mathbb{R} / \mathrm{SO}(m); \mathbb{Q})$ spanned by flat $(m-1)$ -tori grows as one goes up in congruence covers.

INTRODUCTION

Let M be a finite volume, nonpositively curved, locally symmetric manifold. It is usually difficult to determine the homology of such a manifold. However, totally geodesic submanifolds N are natural candidates for non-trivial homology cycles. In this paper we study the case where N is a maximal periodic torus of M . That is N is a compact, totally geodesic, immersed torus whose dimension is equal to the geometric (i.e. real) rank of M . Prasad and Raghunathan have shown [PR72] that a locally symmetric space always contains such tori, while Pettet and Souto have shown that these tori are “stuck” in the thick part of the locally symmetric space and cannot be homotoped out to the end [PS09]. This leads one to suspect that such tori should be homologically nontrivial in a strong sense. The main goal of this paper is to partially justify such suspicions in the special case when $\Gamma < \mathrm{SL}_m \mathbb{Z}$ is a finite index torsionfree subgroup and $M = \Gamma \backslash \mathrm{SL}_m \mathbb{R} / \mathrm{SO}(m)$ is the corresponding locally symmetric space. In this case, maximal periodic tori can be obtained in the following concrete way.

Let $\tau \in \mathrm{SL}_m \mathbb{Q}$ be an element whose characteristic polynomial is irreducible and has m distinct real eigenvalues. The miniset of τ acting on $H := \mathrm{SL}_m \mathbb{R} / \mathrm{SO}(m)$ is a totally geodesic $(m-1)$ -dimensional flat X whose image in the quotient space $\mathrm{SL}_m \mathbb{Z} \backslash \mathrm{SL}_m \mathbb{R} / \mathrm{SO}(m)$ is an isometrically immersed $(m-1)$ -dimensional torus. We show that such tori yield interesting homology cycles in finite volume quotients of H .

Theorem 1. *Let X be an $(m-1)$ -dimensional flat whose image in $M := \mathrm{SL}_m \mathbb{Z} \backslash \mathrm{SL}_m \mathbb{R} / \mathrm{SO}(m)$ is compact. Then, there is a finite cover M' of M such that the image of X in M' is a non-trivial homology cycle in $H_{m-1}(M'; \mathbb{Q})$.*

The key ideas of the proof of this theorem are the following.

- 1) First we find a totally geodesic copy Y of $(\mathrm{SL}_{m-1} \mathbb{R} / \mathrm{SO}(m-1)) \times \mathbb{R}$ in H which is defined over \mathbb{Q} and intersects the flat X transversally (not necessarily orthogonally) in a single point. This reduces to showing that the boundaries at infinity of X and Y are linked.
- 2) Then we find a finite index subgroup $\Gamma \leq \mathrm{SL}_m \mathbb{Z}$ such that the images of X and Y are embedded orientable submanifolds of $\Gamma \backslash \mathrm{SL}_m \mathbb{R} / \mathrm{SO}(m)$ intersecting transversally, with all intersection points having the same sign.

Then the image of X defines a non-trivial cycle in the homology group $H_{m-1}(\Gamma \backslash \mathrm{SL}_m \mathbb{R} / \mathrm{SO}(m); \mathbb{Q})$ and the image of Y defines a non-trivial cycle in homology with closed supports.

Remark 1. The following remains an interesting open question: Is the flat torus (the projection of the flat X to M') in the image of the map $H_{m-1}(\partial M'; \mathbb{Q}) \rightarrow H_{m-1}(M'; \mathbb{Q})$ from the homology of the Borel-Serre boundary of the locally symmetric space to the homology of the interior? A negative answer to this question would give a new proof of the Pettet-Souto result.

Let Γ be a finite index torsion free subgroup of $\mathrm{SL}_m \mathbb{Z}$ and $\Gamma(p^n) := \Gamma \cap \ker(\mathrm{SL}_m \mathbb{Z} \rightarrow \mathrm{SL}_m(\mathbb{Z}/p^n))$ the p^n congruence subgroup. The argument sketched above can be generalized to one that uses multiple flats. We prove the following theorem. It shows that the subspace of homology generated by flat tori grows as one goes up in congruence covers.

Theorem 2. *Given a prime p and an integer N , there is n_0 such that for $n \geq n_0$, the dimension of the subspace of $H_{m-1}(\Gamma(p^n) \backslash H; \mathbb{Q})$ spanned by flat cycles is $\geq N$.*

Remark 2. This also implies nonvanishing for homology in dimensions other than $m-1$. For instance, if $m=3$ then $\Gamma(p^n) \backslash H$ is homotopy equivalent to a 3-complex, $b_1(\Gamma(p^n)) = 0$ by the normal subgroup theorem and $\chi(\Gamma(p^n)) = 0$, hence $b_3(\Gamma(p^n)) = 1 + b_2(\Gamma(p^n))$ grows as one goes up in congruence covers.

Related work. There is a large and fruitful literature on homology of locally symmetric spaces. The reader is referred to the survey article [Sch10] for a discussion of various points of view on the subject and an extensive bibliography. Our work is closest in spirit to [MR81, RS93, LS86]. The idea of eliminating unwanted intersections by passing to congruence covers appears in some form in all these works. However, the homology studied in those papers comes from cycles which are the fixed point sets of a finite order rational isometry σ . One finds another finite order rational isometry σ' commuting with σ and then intersects the fixed point sets. The resulting fixed point sets intersect orthogonally. The flat \mathbb{T}^{m-1} -cycles considered in this paper are not of this type. They are not fixed by any finite order isometry. (The flat $X = \mathbb{R}^{m-1}$ in the universal cover is the fixed set of an abelian group of involutions $(\mathbb{Z}/2)^m$, but these involutions are not rational and do not descend to a finite cover.) Further, our complementary subspaces Y do not need to intersect X orthogonally. This gives flexibility in the choice of Y and allows us to find appropriate intersection patterns in the universal cover via a density argument. The rationally defined subspaces Y which we intersect with the flats X are more familiar. They are studied for instance in [AB90, LS86].

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1. THE SPHERE AT INFINITY OF $\mathrm{SL}_m \mathbb{R} / \mathrm{SO}(m)$

In this section we describe the sphere at infinity of the symmetric space $\mathrm{SL}_m \mathbb{R} / \mathrm{SO}(m)$ in terms of flag-eigenvalue pairs in \mathbb{R}^m (section 2.13.8 in [Ebe96]). Points on the sphere at infinity correspond to geodesic rays e^{tZ} where Z is a trace zero, symmetric matrix of length one. That is, the eigenvalues $\lambda_1 \geq \dots \geq \lambda_m$ of Z arranged in descending order satisfy

$$\begin{aligned} \lambda_1 + \dots + \lambda_m &= 0, \\ \lambda_1^2 + \dots + \lambda_m^2 &= 1. \end{aligned}$$

Let $E_i := \{v \in \mathbb{R}^m \mid Zv = \lambda_i v\}$ be the λ_i -eigenspace of Z . The eigenvalues together with the flag

$$0 \subset E_1 \subset E_1 + E_2 \subset \dots \subset E_1 + \dots + E_{m-1} \subset \mathbb{R}^m$$

provide enough information to recover the symmetric matrix Z . Thus, the points on the sphere at infinity are parametrized by flag-eigenvalue pairs. The flags are best thought of as nested arrangements of points, lines, planes etc in real projective space \mathbb{P}^{m-1} .

The sphere of a direct sum decomposition. Suppose we are given a direct sum decomposition $\mathbb{R}^m = U_1 \oplus \dots \oplus U_r$. Up to finite index, the subgroup of $\mathrm{GL}(\mathbb{R}^m)$ preserving this decomposition is $\mathrm{GL}(U_1) \times \dots \times \mathrm{GL}(U_r)$ and the subgroup of $\mathrm{SL}(\mathbb{R}^m)$ preserving the decomposition is $\mathbb{R}^{r-1} \times \mathrm{SL}(U_1) \times \dots \times \mathrm{SL}(U_r)$. This group acts on $\mathrm{SL}(\mathbb{R}^m) / \mathrm{SO}(\mathbb{R}^m)$ preserving a totally geodesic symmetric subspace of the form

$$(1) \quad \mathbb{R}^{r-1} \times \mathrm{SL}(U_1) / \mathrm{SO}(U_1) \times \dots \times \mathrm{SL}(U_r) / \mathrm{SO}(U_r).$$

We denote the sphere at infinity of this symmetric subspace by $S(U_1, \dots, U_r)$. We note that $S(\mathbb{R}) = \emptyset$ because $\mathrm{SL}(\mathbb{R})$ is a point. Generally, if U is an n -dimensional vector space then $S(U) = S^{n(n+1)/2-2}$. The product decomposition (1) gives a join decomposition

$$(2) \quad S(U_1, \dots, U_r) = S^{r-2} \star S(U_1) \star \dots \star S(U_r)$$

for the sphere at infinity. Next, we describe the flags that occur on this sphere.

The sphere $S(U_1, \dots, U_r)$ as the fix set of an element in $\mathrm{SL}(\mathbb{R}^m)$. Let $\tau \in \mathrm{SL}(\mathbb{R}^m)$ be a diagonalizable element whose eigenspace decomposition is $\mathbb{R}^m = U_1 \oplus \dots \oplus U_m$. The minset of τ is the symmetric subspace $\mathbb{R}^{r-1} \times \mathrm{SL}(U_1) / \mathrm{SO}(U_1) \times \dots \times \mathrm{SL}(U_r) / \mathrm{SO}(U_r)$. The action of τ extends to the sphere at infinity $S(\mathbb{R}^m)$ and its fixed set is precisely the sphere $S(U_1, \dots, U_r)$. The action of τ on the sphere at infinity does not change the eigenvalues and sends a flag $F_1 \subset \dots \subset F_k$ to the flag $\tau F_1 \subset \dots \subset \tau F_k$ (see 2.13.8 of [Ebe96].) The element τ also acts on the projective space \mathbb{P}^{m-1} . The subspaces U_i form a transverse arrangement (U_1, \dots, U_r) of projective subspaces in \mathbb{P}^{m-1} . A subspace $V \cong \mathbb{P}^k \subset \mathbb{P}^{m-1}$ is *associated* to the arrangement (U_1, \dots, U_r) if there are $(k+1)$ points in the union $\bigcup_{i=1}^r U_i$ that span V . A flag is associated to the arrangement (U_1, \dots, U_r) if all the subspaces in the flag are associated to (U_1, \dots, U_r) . It is not hard to see that the flags associated to (U_1, \dots, U_r) are precisely the flags preserved

¹Emphasis added to avoid confusion experienced by referees of the second author's previous papers.

by τ . Consequently, the sphere at infinity $S(U_1, \dots, U_r)$ consists of those flag-eigenvalue pairs whose flag is associated to (U_1, \dots, U_r) .

Intersecting spheres at infinity. Suppose we have two direct sum decompositions

$$L_1 \oplus \dots \oplus L_m \cong \mathbb{R}^m \cong L \oplus P$$

where L, L_1, \dots, L_m are points and P is a hyperplane in \mathbb{P}^{m-1} . If L_1, \dots, L_m, L and P are in general position, then the only subspaces associated to both $(L_1, \dots, L_{m-2}, \overline{L_{m-1}L_m})$ and (L, P) are the point $L' := P \cap \overline{L_{m-1}L_m}$ and the hyperplane $Q := \overline{LL_1 \dots L_{m-2}}$. Here, we denote by $\overline{L_{m-1}L_m}$ the projective line passing through the points L_{m-1} and L_m etc. By general position the point L' is not contained in the hyperplane Q . Thus, the spheres at infinity $S(L_1, \dots, L_{m-2}, \overline{L_{m-1}L_m})$ and $S(L, P)$ intersect at exactly two points L' and Q in $S(\mathbb{R}^m)$.

A neighborhood of Q . The singleton codimension one flags form a projective space \mathbb{P}_{m-2}^{m-1} in the sphere at infinity $S(\mathbb{R}^m)$. This projective space has a regular neighborhood $N(\mathbb{P}_{m-2}^{m-1})$ in $S(\mathbb{R}^m)$ which is a bundle whose fibre over a point $V \in \mathbb{P}_{m-2}^{m-1}$ can be identified with the cone $\text{Cone}(S(V))$ of the sphere at infinity of V .

2. THE LINKING LEMMA

In this section we explain how to compute intersections of totally geodesic submanifolds in the symmetric space $\text{SL}_m \mathbb{R} / \text{SO}(m)$ in terms of linking on the sphere at infinity. We then describe how to determine linking at infinity in terms of the geometry of real projective space. This description turns out to be convenient for constructing and perturbing intersection patterns in the universal cover.

Suppose that $X \cong \mathbb{R}^{m-1}$ is a flat obtained as the minset of an element $\tau \in \text{SL}_m \mathbb{R}$ with m distinct real eigenvalues, while Y is a copy of $(\text{SL}_{m-1} \mathbb{R} / \text{SO}(m-1)) \times \mathbb{R}$ which is the minset of an involution $\rho \in \text{GL}_m \mathbb{R}$ with eigenvalues $(-1, \dots, -1, 1)$. Let (L_1, \dots, L_m) be the eigenspaces of τ and (L, P) be the line-hyperplane pair of eigenspaces of ρ . Suppose that L_1, \dots, L_m, L and P are in general position. Then the spheres at infinity $\partial X = S(L_1, \dots, L_m)$ and $\partial Y = S(L, P)$ are disjoint. Using geodesic projection through a point to the sphere at infinity $S(\mathbb{R}^m)$ one sees that X and Y intersect if and only if the spheres ∂X and ∂Y link in $S(\mathbb{R}^m)$. If there is an intersection, then it is necessarily transverse because the spheres ∂X and ∂Y are disjoint.

We give a geometric criterion for determining when the spheres ∂X and ∂Y link. Denote by V_i the hyperplane which passes through all the points $L_1, \dots, L_m \in \mathbb{P}^{m-1}$ except L_i .

Proposition 3. *Suppose that (L_1, \dots, L_m) and (L, P) are in general position. The hyperplanes V_1, \dots, V_m subdivide \mathbb{P}^{m-1} into open $(m-1)$ -simplices. The spheres $S(L_1, \dots, L_m)$ and $S(L, P)$ link if and only if the $(m-1)$ -simplex σ containing L does not meet P .*

Proof. We do an inductive argument. The base case $m = 2$ is easy. In this case the proposition is simply identifying the projective line \mathbb{P}^1 with the circle at infinity $\partial \mathbb{H}^2$.

Now, we suppose that the Proposition is known for $m-1$ and prove it for m .

- We begin with the following observation. If P meets the simplex σ , then it intersects at least one of the edges of the simplex σ . Without loss of generality, it intersects the edge E connecting the vertices L_{m-1} and L_m . Now, let $Q := \overline{LL_1 \dots L_{m-2}}$ be the

hyperplane passing through the points L, L_1, \dots, L_{m-2} . Since L is contained inside the simplex σ , the hyperplane Q also intersects the edge E . Thus, the points $\{L_{m-1}, L_m\}$ do not link the points $\{P \cap \overline{L_{m-1}L_m}, Q \cap \overline{L_{m-1}L_m}\}$ on the line $\overline{L_{m-1}L_m}$.

- The pair of points $\{L_{m-1}, L_m\} = S(L_{m-1}, L_m) = S^0$ lies inside the circle $S(\overline{L_{m-1}L_m}) = S^1$. It links the points $\{P \cap \overline{L_{m-1}L_m}, Q \cap \overline{L_{m-1}L_m}\}$ in this circle if and only if the suspension $S(L_1, \dots, L_m) = S^{m-3} \star S(L_{m-1}, L_m)$ links the points $\{P \cap \overline{L_{m-1}L_m}, Q \cap \overline{L_{m-1}L_m}\}$ in the suspension $S(L_1, \dots, L_{m-2}, \overline{L_{m-1}L_m}) = S^{m-3} \star S(\overline{L_{m-1}L_m})$. This is the same as linking the points $\{P \cap \overline{L_{m-1}L_m}, Q\}$ because the flags Q and $Q \cap \overline{L_{m-1}L_m}$ are “adjacent” (the segment on the sphere at infinity corresponding to the flag $Q \cap \overline{L_{m-1}L_m} \subset Q$ connects them and does not meet $S(L_1, \dots, L_m)$) so they lie in the same component of $S(L_1, \dots, L_{m-2}, \overline{L_{m-1}L_m}) \setminus S(L_1, \dots, L_m)$. In summary, we have proved the following lemma.

Lemma 4. *The sphere $S(L_1, \dots, L_m)$ links the pair of points $\{P \cap \overline{L_{m-1}L_m}, Q\}$ inside $S(L_1, \dots, L_{m-2}, \overline{L_{m-1}L_m})$ if and only if P does not meet the simplex σ .*

- To unburden notation slightly, we will from now on denote the sphere $S(L_1, \dots, L_m)$ by the letter $S := S(L_1, \dots, L_m)$.

Since (L_1, \dots, L_m) and (L, P) are in general position, the spheres at infinity $S(L, P)$ and $S(L_1, \dots, L_{m-2}, \overline{L_{m-1}L_m})$ intersect in precisely the two points $P \cap \overline{L_{m-1}L_m}$ and Q . If P meets the simplex σ then by Lemma 4 one of the two connected components of $S(L_1, \dots, L_{m-2}, \overline{L_{m-1}L_m}) \setminus S$ is a ball which is bounded by S and does not intersect $S(L, P)$. This means that the sphere S does not link the sphere $S(L, P)$ which is half of what we needed to show.

- Now, suppose that P does not meet the simplex σ . It remains to show that in this case the sphere S links the sphere $S(L, P)$. By Lemma 4, in this situation one of the components of $S(L_1, \dots, L_{m-2}, \overline{L_{m-1}L_m}) \setminus S$ contains Q but does not contain $P \cap \overline{L_{m-1}L_m}$. We call this component D . We have the ball D with boundary $\partial D = S$. The sphere $S(L, P)$ does not meet S and intersects the ball D in a single point Q . To show that S and $S(L, P)$ link, it is enough to show that the intersection at Q is transverse.
- Recall that Q is contained in the projective space \mathbb{P}_{m-2}^{m-1} of singleton codimension one flags on the sphere at infinity $S(\mathbb{R}^m)$. This projective space has a regular neighborhood $N(\mathbb{P}_{m-2}^{m-1})$ which is a bundle whose fibre over a point $V \in \mathbb{P}_{m-2}^{m-1}$ can be identified with the cone $\mathrm{Cone}(S(V))$ of the sphere at infinity of V .

Thus to show the intersection of D and $S(L, P)$ at Q is transverse, it suffices to check that the restriction of the intersection to \mathbb{P}_{m-2}^{m-1} is transverse at Q and that the spheres $D \cap S(Q) = S(L_1, \dots, L_{m-2}, Q \cap \overline{L_{m-1}L_m})$ and $S(L, P) \cap S(Q) = S(L, Q \cap P)$ link in $S(Q)$. We will do this in the following two bullets. This will conclude the proof of the Proposition.

- First, $D \cap \mathbb{P}_{m-2}^{m-1}$ is a neighborhood of Q in a projective line in \mathbb{P}_{m-2}^{m-1} (the line of all hyperplanes passing through the points L_1, \dots, L_{m-2} in \mathbb{P}_1^{m-1}) together with a finite set of $V_i \in \mathbb{P}_{m-2}^{m-1}$ while $S(L, P) \cap \mathbb{P}_{m-2}^{m-1}$ is a neighborhood of Q in a projective hyperplane in \mathbb{P}_{m-2}^{m-1} (the hyperplane of all hyperplanes passing through L in \mathbb{P}_1^{m-1}) together with the single point $P \in \mathbb{P}_{m-2}^{m-1}$. By general position, the line and hyperplane intersect transversally at Q .

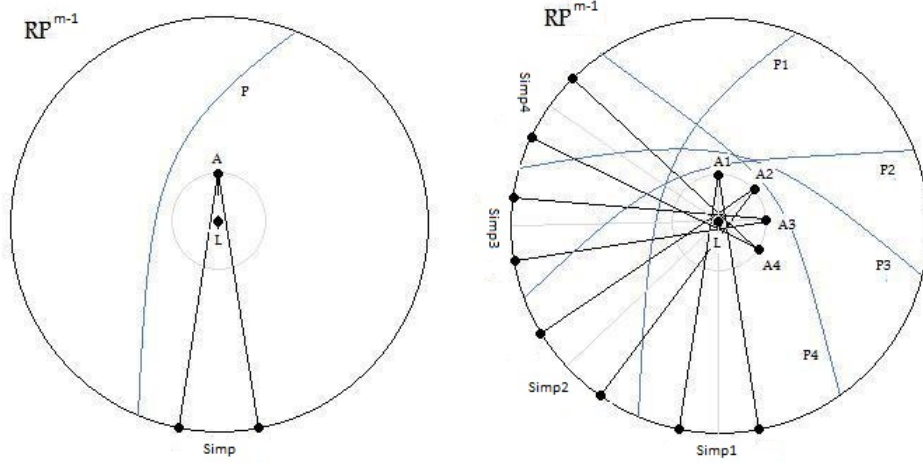


FIGURE 1.

- Second, recall that L is contained in the $(m-1)$ -simplex σ , the hyperplane P does not meet σ , and the hyperplane $Q = \overline{LL_1 \cdots L_{m-2}}$ passes through the points L, L_1, \dots, L_{m-2} . Now, we intersect with the hyperplane $Q \cong \mathbb{P}^{m-2}$. Notice that L is contained in the $(m-2)$ -simplex $Q \cap \sigma$ and the hyperplane $Q \cap P$ in Q does not meet $Q \cap \sigma$. Further, the simplex $Q \cap \sigma$ has vertices $L_1, \dots, L_{m-2}, Q \cap \overline{L_{m-1}L_m}$. Thus, we can apply the inductive hypothesis to conclude the spheres $S(L_1, \dots, L_{m-2}, Q \cap \overline{L_{m-1}L_m})$ and $S(L, Q \cap P)$ link in $S(Q)$.

□

2.1. An arrangement of intersections. Using Proposition 3 it is easy to construct a pattern $X_1, \dots, X_N, Y_1, \dots, Y_N$ of flats X_i and copies of $(\mathrm{SL}_{m-1} \mathbb{R} / \mathrm{SO}(m-1)) \times \mathbb{R}$ denoted Y_i , for which X_i intersects Y_j if and only if $i \leq j$. One such pattern (with $N = 4$) is indicated in Figure 1. We draw the ball model of projective space $\mathbb{R}P^{m-1}$, with points on the boundary of the ball identified via the antipodal map. The right picture is obtained from the left picture via rotations by a fixed amount. To get the required pattern for a general N one starts with a sufficiently thin geodesic $(m-1)$ -simplex $A \star \text{Simp}$ and uses a small enough rotation.

3. ELEMENTS DEFINED OVER \mathbb{Q}

Everything we've done so far has been in the symmetric space $\mathrm{SL}_m \mathbb{R} / \mathrm{SO}(m)$. The rational structure has not yet entered the picture. It starts to play a role when one tries to understand how the spaces X and Y project to arithmetic quotients of the symmetric space.

Rational flats. Let $\tau \in \mathrm{SL}_m \mathbb{Q}$ be an element with m distinct real eigenvalues, and denote its minset by X . It is a totally geodesic submanifold of H . The centralizer $C_\tau(\mathbb{R}) \cong (\mathbb{R}^*)^{m-1}$ acts transitively on the minset by orientation preserving isometries. The group of all isometries

preserving the minset $S_X(\mathbb{R}) := \{g \in \mathrm{SL}_m \mathbb{R} \mid gX = X\}$ is the semidirect product $C_\tau(\mathbb{R}) \rtimes S_m$ of the centralizer with the symmetric group on m letters (the symmetric group permutes the eigenspaces of τ .) Let $\Gamma < \mathrm{SL}_m \mathbb{Z}$ be a finite index torsionfree subgroup and denote by $\Gamma_X := \Gamma \cap S_X(\mathbb{R})$ the group of all isometries in Γ preserving the flat X . The following lemma shows that after passing to a deep enough congruence subgroup we can assume that all isometries of Γ_X commute with τ .

Lemma 5. *Fix a prime p . Then $\Gamma_X(p^n) \subset C_\tau(\mathbb{R})$ for sufficiently large n .*

Proof. Let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of τ and $K := \mathbb{Q}(\lambda_1, \dots, \lambda_m)$ the field obtained by adjoining those eigenvalues. The group $S_X(K)$ is $\mathrm{SL}_m K$ -conjugate to $(K^*)^{m-1} \rtimes S_m$, where the $(K^*)^{m-1}$ consists of determinant one diagonal matrices and the symmetric group S_m is represented by permutation matrices. From this it follows that $S_X(K)$ decomposes into $C_\tau(K)$ -cosets

$$S_X(K) = C_\tau(K) \cup C_\tau(K)\gamma_1 \cup \dots \cup C_\tau(K)\gamma_r,$$

with the matrices γ_i being conjugates of the permutation matrices, i.e. lying in $\mathrm{SL}_m K$. Let $K_p := \mathbb{Q}_p(\lambda_1, \dots, \lambda_m)$ be the p -adic completion. Note that the non-identity cosets lie in the closed subset

$$\bigcup_{i=1}^r \{x \in \mathrm{SL}_m K_p \mid [x\gamma_i^{-1}, \tau] = 1\},$$

of $\mathrm{SL}_m K_p$. This subset does not contain 1 since $[\gamma_i^{-1}, \tau] \neq 1$, so there is a small p -adic neighborhood of the identity $U(p^n)$ where all elements from $S_X(K)$ commute with τ i.e.

$$\Gamma_X(p^n) = \Gamma_X \cap U(p^n) \subset S_X(K) \cap U(p^n) \subset C_\tau(\mathbb{R}).$$

□

Since τ is a matrix with entries in \mathbb{Q} , the image of X in H/Γ is an isometrically immersed $(m-1)$ -dimensional flat. (See Theorem D in [Sch10] for a proof of this.) Let $p_\tau(t) = \det(t - \tau)$ be the characteristic polynomial. The following is a special case of a theorem of Prasad and Raghunathan in [PR72]:

Proposition 6. *Suppose $\tau \in \mathrm{SL}_m \mathbb{Q}$ has m distinct real eigenvalues and irreducible characteristic polynomial. Then X/Γ_X is compact and finitely covered by a $(m-1)$ -dimensional torus \mathbb{T}^{m-1} .*

Rational copies of $(\mathrm{SL}_{m-1} \mathbb{R} / \mathrm{SO}(m-1)) \times \mathbb{R}$. Now, let $\rho \in \mathrm{GL}_m \mathbb{Q}$ be a diagonalizable matrix with eigenvalues $(-1, \dots, -1, 1)$. Then the centralizer $C_\rho(\mathbb{R}) \cong \mathrm{GL}_{m-1} \mathbb{R}$ acts on the minset $Y \cong \mathrm{SL}_{m-1} \mathbb{R} / \mathrm{SO}(m-1) \times \mathbb{R}$ of ρ , and the quotient Y/Γ_Y is a properly immersed submanifold of H/Γ . (Theorem D in [Sch10].) In this case the group $S_Y(\mathbb{R})$ of isometries preserving the flat Y is just the centralizer $C_\rho(\mathbb{R})$. Note that $C_\rho(\mathbb{R}) \cong \mathrm{GL}_{m-1} \mathbb{R}$ has two components. If m is even, then the entire centralizer preserves the orientation of Y , but if m is odd, then the elements not in the identity component do not preserve the orientation of Y . The eigenspaces of ρ are a (rational!) line L and a hyperplane P . Whether or not an element preserves orientation can be determined by its action on the line L . (This is noted in the discussion after Corollary 2.4 of [LS86].) Let $\gamma \in C_\tau(\mathbb{Z})$ be an element in the centralizer with integer entries. Then $1 = \det(\gamma) = \det(\gamma|_L) \det(\gamma|_P)$ and since γ has integer entries we must have $\gamma|_L = \pm 1$. Further, the element γ preserves orientations precisely when we have

+ signs, i.e. $\det \gamma|_P = \det \gamma|_L = 1$. Let $0 \neq v \in L_{\mathbb{Z}} \subset \mathbb{Z}^m \subset \mathbb{Q}^m$ be a non-zero vector with integer entries. Then, $\gamma v = \pm v$. For sufficiently large n we have $v \neq -v$ in the quotient $L_{\mathbb{Z}/p^n} \subset (\mathbb{Z}/p^n)^m$ so that the subgroup $\Gamma_Y(p^n)$ which acts trivially on $L_{\mathbb{Z}/p^n}$ must preserve the orientation of Y .

Embeddings vs immersions. A result of Raghunathan (Theorem E in [Sch10]) shows that we can always find a positive integer K_0 such that for $K \geq K_0$ the maps $X/\Gamma_X(K) \rightarrow H/\Gamma(K)$ and $Y/\Gamma_Y(K) \rightarrow H/\Gamma(K)$ are embeddings. Thus, we can replace the group Γ by the congruence subgroup $\Gamma(p^n)$, $p^n \geq K_0$ to make sure that the quotients $X/\Gamma_X(p^n)$ and $Y/\Gamma_Y(p^n)$ are embedded in the quotient $H/\Gamma(p^n)$. We remark that this is the same as saying that $\Gamma(p^n)X$ is a disjoint union of copies of X in H , and $\Gamma(p^n)Y$ is a disjoint union of copies of Y in H .

We summarize the conclusions of this section in the following proposition. The notations are the same as the ones used throughout the section.

Proposition 7. *Let p be a prime. Then for sufficiently large n ,*

- $\Gamma(p^n)X$ is a disjoint union of copies of X and $\Gamma(p^n)Y$ is a disjoint union of copies of Y .
- The subgroup $\Gamma_X(p^n)$ centralizes τ and the subgroup $\Gamma_Y(p^n)$ centralizes ρ .
- There are $\Gamma(p^n)$ -invariant orientations on $\Gamma(p^n)X$ and $\Gamma(p^n)Y$.

Proof. Everything except for the third point has already been proved in the section above. Further, we've shown that $\Gamma_Y(p^n)$ preserves an orientation on Y for large enough n . For such n we can—starting with an orientation Y^+ of Y —define a $\Gamma(p^n)$ -invariant orientation on $\Gamma(p^n)Y$ by $(\gamma Y)^+ = \gamma Y^+$ for $\gamma \in \Gamma_Y(p^n)$.

For large enough n , the group $\Gamma_X(p^n)$ is contained in the centralizer $C_\tau(\mathbb{R})$, and this centralizer preserves an orientation on X . Hence, we can define an invariant orientation on $\Gamma(p^n)X$ in the same way as for Y . \square

4. RATIONAL INTERSECTION PATTERNS

In subsection 2.1 we described a certain intersection pattern $\{X_i, Y_i\}_{i=1}^N$ involving finitely many flats and copies $(\mathrm{SL}_{m-1} \mathbb{R} / \mathrm{SO}(m-1)) \times \mathbb{R}$. In this section we explain how to get the same intersection pattern using rational flats that are compact in the quotient, and copies of $(\mathrm{SL}_{m-1} \mathbb{R} / \mathrm{SO}(m-1)) \times \mathbb{R}$ that are defined over \mathbb{Q} .

Intersection patterns are preserved by small perturbations. From the description of intersections between X_i and Y_i given Proposition 3, it is clear that an intersection pattern does not change when X_i is perturbed to a nearby flat X'_i , (so the m -tuple of eigenspaces is perturbed slightly). Similarly, the intersection pattern does not change when Y_i is perturbed to a nearby Y'_i (so the line-hyperplane pair (L, P) is perturbed slightly.)

Small rational perturbations of X and Y exist. Note that the $\mathrm{SL}_m \mathbb{Q}$ -orbit of transverse triple (L_1, \dots, L_m) is dense in the space of all such triples. In particular let τ be an element with m distinct real eigenvalues and irreducible characteristic polynomial. The orbit of the m -tuple of eigenspaces of τ is dense. Thus, any flat X has arbitrarily small perturbations whose quotients are compact $(m-1)$ -dimensional tori. Similarly, the rational line-hyperplane pairs are dense in the space of all such pairs, so Y has arbitrarily small perturbations whose quotients are properly immersed.

Putting these two observations together with the intersection pattern described in subsection 2.1, we get the following

Proposition 8. *There are rational flats $\{X_i\}_{i=1}^N$ whose quotients X_i/Γ_{X_i} are compact and rational copies of $(\mathrm{SL}_{m-1} \mathbb{R} / \mathrm{SO}(m-1)) \times \mathbb{R}$ denoted $\{Y_i\}_{i=1}^N$ whose quotients Y_i/Γ_{Y_i} are properly immersed in H/Γ such that X_i intersects Y_j if and only if $i \leq j$. Further, all the intersections of the X_i and the Y_j are transverse.*

5. PUSHING INTERSECTION PATTERNS DOWN TO A CONGRUENCE COVER

The goal of this section is to show that the intersection pattern described in Proposition 8 can be pushed down to sufficiently deep congruence covers. Once this is done, we will be able to conclude that the submanifolds involved in the intersection pattern give nontrivial homology cycles in those congruence covers.

Theorem 9. *Let p be a prime. Suppose that $\tau \in \mathrm{SL}_m \mathbb{Q}$ is a matrix with m distinct real eigenvalues and irreducible characteristic polynomial, while $\rho \in \mathrm{GL}_m \mathbb{Q}$ is a diagonalizable matrix with eigenvalues $(-1, \dots, -1, 1)$. Let X and Y be the minsets of τ and ρ , respectively. Suppose further that $C_\rho(\mathbb{R}) \cap C_\tau(\mathbb{R}) = 1$.*

- (1) *If X and Y intersect transversally in a single point $z \in H$, then for sufficiently large n the quotients $X/\Gamma_X(p^n)$ and $Y/\Gamma_Y(p^n)$ are orientable, all their intersections in $H/\Gamma(p^n)$ are transverse and have the same sign.*
- (2) *If X and Y are disjoint then for sufficiently large n the quotients $X/\Gamma_X(p^n)$ and $Y/\Gamma_Y(p^n)$ are disjoint in $H/\Gamma(p^n)$.*

PROOF OF THEOREM 9

- Proposition 7 shows that, replacing Γ by a sufficiently deep congruence subgroup $\Gamma(p^n)$ if necessary, we may assume ΓX is a disjoint union of X , equipped with a Γ -invariant orientation (so that if X^+ is the oriented flat then $(\gamma X)^+ = \gamma X^+$ for all $\gamma \in \Gamma$) and $\Gamma_X \subset C_\tau(\mathbb{R})$, i.e. all elements of Γ preserving the flat X commute with the element τ . Similarly for the subspace Y .
- The quotient X/Γ_X is compact (by Proposition 6), so the intersections $\Gamma X \cap Y$ break up into finitely many Γ_Y -orbits $\{\Gamma_Y \gamma_i X \cap Y\}_{i=0}^n$. We look at the double cosets $\Gamma_Y \gamma_i \Gamma_X$.
- If $\gamma \notin C_\rho(\mathbb{Z}_p)C_\tau(\mathbb{Z}_p)$ then for large enough n we have $\Gamma(p^n)\gamma \cap \Gamma_Y \Gamma_X = \emptyset$ which implies that $\Gamma_Y \gamma \Gamma_X \cap \Gamma(p^n) = \emptyset$. In other words, the flats $\Gamma_Y \gamma X$ don't occur in $\Gamma(p^n)X$. Thus, after replacing Γ with a deep enough congruence subgroup $\Gamma(p^n)$ we may assume that $\gamma_i \in C_\rho(\mathbb{Z}_p)C_\tau(\mathbb{Z}_p)$ for all i .
- The condition $C_\rho(\mathbb{R}) \cap C_\tau(\mathbb{R}) = 1$ implies the only solutions x to the system of equations $x\rho = \rho x, x\tau = \tau x$ are scalar matrices.

Proof. The set of solutions is a vector subspace in the space $\mathrm{End}(\mathbb{R}^m)$ of real $m \times m$ matrices. Thus, if there is a solution that is not a scalar matrix, then there is also a non-scalar solution x' of positive determinant $\det(x') > 0$ (pick x' close enough to the identity matrix). We rescale to get a non-identity matrix $x' \cdot (\det(x'))^{-1/m} \in C_\rho(\mathbb{R}) \cap C_\tau(\mathbb{R})$. This gives a contradiction. \square

Since this is true over \mathbb{Q} , it is also true over \mathbb{Q}_p and consequently $C_\rho(\mathbb{Q}_p) \cap C_\tau(\mathbb{Q}_p) = 1$. Hence, the map

$$\begin{aligned} C_\rho(\mathbb{Z}_p) \times C_\tau(\mathbb{Z}_p) &\rightarrow C_\rho(\mathbb{Z}_p)C_\tau(\mathbb{Z}_p), \\ (x, y) &\mapsto xy, \end{aligned}$$

is a continuous bijection. Since the spaces on both sides are compact Hausdorff, it is a homeomorphism (but not a group homomorphism: the space on the right isn't even a group). If $\gamma_i = a'b'$ with $a' \in C_\rho(\mathbb{Z}_p)$ and $b' \in C_\tau(\mathbb{Z}_p)$ then we can embed the double coset $\Gamma_Y \gamma_i \Gamma_X$ into the product via

$$\begin{aligned} \Gamma_Y \gamma_i \Gamma_X &\hookrightarrow C_\rho(\mathbb{Z}_p) \times C_\tau(\mathbb{Z}_p) \\ g\gamma_i h &\mapsto (ga', b'h) \end{aligned}$$

for $g \in \Gamma_Y, h \in \Gamma_X$. The intersection corresponding to a double coset $\Gamma_Y \gamma_i \Gamma_X$ can be removed by passing to a p^n congruence cover precisely when the double coset misses a neighborhood of the identity in this product. This description will be useful below.

- The following linear algebraic lemma lets us translate from \mathbb{Z}_p to \mathbb{Q} .

Lemma 10. *If $\gamma \in C_\rho(\mathbb{Z}_p)C_\tau(\mathbb{Z}_p)$ then there are rational matrices $a, b \in \mathrm{GL}_m \mathbb{Q}$, $\gamma = ab$, $[a, \rho] = 1$, $[b, \tau] = 1$.*

Proof. Suppose that γ can be expressed as a product $\gamma = a'b'$ where $a' \in C_\rho(\mathbb{Z}_p)$ and $b' \in C_\tau(\mathbb{Z}_p)$. The element a' satisfies the equations $[a', \tau] = [\gamma, \tau]$ and $[a', \rho] = 1$. These equations can be rewritten as

$$(3) \quad a'\tau = [\gamma, \tau]\tau a',$$

$$(4) \quad a'\rho = \rho a'.$$

These equations are linear homogeneous in the matrix entries of a' and are defined over \mathbb{Q} (since γ, τ , and ρ are defined over \mathbb{Q}). If they have a p -adic solution a' of determinant one, then they also have a rational solution a with non-zero determinant. Take $b = a^{-1}\gamma$. \square

- The following lemma reduces to the case where γ_i is a product of an element that acts on X in an orientation preserving way with an element that acts on Y in an orientation preserving way.

Lemma 11. *For sufficiently large n , all the flats in $\Gamma(p^n)X$ which intersect Y are of the form $a_i b_i X$ where $a_i \in C_\rho(\mathbb{Q})$, $b_i \in C_\tau(\mathbb{Q})$, a_i preserves the orientation of Y and b_i preserves the orientation of X .*

Proof. If $\gamma_i \in C_\rho(\mathbb{Z}_p)C_\tau(\mathbb{Z}_p)$ then write $\gamma_i = a'b' = ab$ with a', b', a, b as in Lemma 10. The condition $C_\rho(\mathbb{R}) \cap C_\tau(\mathbb{R}) = 1$ means the p -adic solution a' and rational solution a in Lemma 10 are scalar multiples of each other, i.e. $a' = a \cdot (\det(a))^{-1/m}$.

We now look at how the element a acts on the line L . Pick a non-zero vector $0 \neq v \in L_{\mathbb{Q}}$ and note that $av = kv$ for some $k \in \mathbb{Q}$. Also, recall that the group Γ_Y consists of integer matrices preserving the orientation of Y , so that $\Gamma_Y v = v$. Thus

$$\Gamma_Y a' v = \Gamma_Y a \cdot (\det a)^{-1/m} v = k \cdot (\det a)^{-1/m} v.$$

If $k \cdot (\det a)^{-1/m} \neq 1$ then $\Gamma_Y a' \cap U(p^n) = \emptyset$ for a sufficiently small p -adic neighborhood of the identity $U(p^n)$. Thus, the double coset $\Gamma_Y a' b' \Gamma_X = \Gamma_Y \gamma_i \Gamma_X$ misses the neighborhood of the identity $U(p^n) \times C_\tau(\mathbb{Z}_p)$ in the product $C_\rho(\mathbb{Z}_p) \times C_\tau(\mathbb{Z}_p)$. This means it can be eliminated from the intersection by replacing Γ with a sufficiently deep congruence subgroup $\Gamma(p^r)$.

On the other hand, if $k \cdot (\det a)^{-1/m} = 1$ then $k = (\det a)^{1/m}$ is a rational number. Rescaling by k , we may write the element γ as $\gamma = ab = (a/k)(kb)$ with $a/k \in C_\rho(\mathbb{Q})$ and $kb \in C_\tau(\mathbb{Q})$. Note that a/k preserves the orientation of Y since $(a/k)v = v$, and kb preserves the orientation of X since it lies in the centralizer of τ . \square

- We are now in the situation where all intersections $\Gamma(p^n)X \cap Y$ can be written in the form $a_i b_i X^+ \cap Y^+$ for orientation preserving elements $a_i \in C_\rho(\mathbb{Q})$, $b_i \in C_\tau(\mathbb{Q})$. Thus

$$a_i b_i X^+ \cap Y^+ = a_i X^+ \cap Y^+ = a_i (X^+ \cap a_i^{-1} Y^+) = a_i (X^+ \cap Y^+).$$

This means all the intersections are transverse and have the same sign as the intersection $X^+ \cap Y^+$. This proves the first part of the theorem. Further, it means that if X and Y do not intersect, then $a_i b_i X$ and Y do not intersect. This proves the second part of the theorem.

PROOF OF THEOREM 2

The condition $C_\rho(\mathbb{R}) \cap C_\tau(\mathbb{R}) = 1$ is satisfied whenever the collection of eigenspaces (L_1, \dots, L_m) of τ and the line-hyperplane pair (L, P) of ρ are in general position.

Let X_1, \dots, X_N and Y_1, \dots, Y_N be the subspaces obtained in Proposition 8. The eigenspaces of X_i and line-hyperplane pairs of Y_j are in general position, so we can apply Theorem 9 and find n_0 such that for $n \geq n_0$ the quotients $\overline{X}_i = X_i / \Gamma_{X_i}(p^n)$ and $\overline{Y}_i = Y_i / \Gamma_{Y_i}(p^n)$

- are embedded orientable manifolds,
- \overline{X}_i and \overline{Y}_i intersect,
- all the intersections of \overline{X}_i and \overline{Y}_i are transverse and have the same sign,
- \overline{X}_i and \overline{Y}_j do not intersect for $i > j$.

This means the intersection matrix is upper triangular with non-zero diagonal entries. Consequently, it is a non-degenerate $N \times N$ matrix, which means the flats \overline{X}_i span an N -dimensional subspace of $H_2(H/\Gamma(p^n); \mathbb{Q})$.

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